Cartier divisors (Har II 6)

Cartier divisors allow us to extend the notion of divisors to arbitrary schemes.

First, to give some motivation, assume X satisfies the properties from the previous section (integral, Noeth., sep., regular in codim one). Additionally, assume X is normal. In this case, a Cartier divisor is a special kind of Weil divisor, which is "locally principal." More precisely:

Def: If D is a Weil divisor on X normal, D is <u>locally principal</u> if there is an open cover $\{U_i\}$ of X and corresponding $f_i \in K(X)^*$ such that

$$D \cap U_i = (f_i) \quad \text{on } U_i.$$

Note that since on $U_i \cap U_j$, $(f_i) = (f_j)$, we get $(f_i/f_j) = 0$ on $U_i \cap U_j$. i.e. $v_Y(f_i/f_j) = 0$ for all Y whose generic point is in $U_i \cap U_j$.

This means that for any open $\operatorname{Spec} A \subseteq U_i \cap U_j$, f_i/f_j is a unit in every A_P, P codim 1. Thus, $f_i/f_j \in \bigcap_{\operatorname{codim} P = 1} A_P = A$ by normality.

Thus, f_i/f_j is regular on U. Similarly, f_j/f_i is as well, so f_i/f_j is a unit in $\mathcal{O}_X(U)$. That is,

$$f_i/f_i \in \mathcal{O}_X^*(U),$$

where $\mathcal{O}_X^*$ is the sheaf on X defined

$$U \longmapsto \mathcal{O}_X(U)^* = \text{group of units (w/ mult.)}$$

We use this characterization to define Cartier divisors, a generalization of locally principal divisors to an arbitrary scheme.

<u>Def:</u> X a scheme. For each affine open $U = \operatorname{Spec} A$, let $S \subseteq A$ be the nonzero divisors. The <u>total quotient ring</u> of A is $K(U) = S^{-1}A$. For any open U, define

$$S(U) = \left\{ r \in \Gamma(U, \mathcal{O}_X) \;\middle|\; \begin{array}{l} r \text{ is a NZD in } \mathcal{O}_x \\ \text{for all } x \in U \end{array} \right\}.$$

(check: this is the same as S above when $U = \operatorname{Spec} A$). The rings

$$S(U)^{-1}\Gamma(U, \mathcal{O}_X)$$

form a presheaf, and the sheafification is $\mathcal{K}$, called the <u>sheaf of total quotient rings of $\mathcal{O}$</u>.

Note that if X is integral, $\mathcal{K}(U) = K(X)$ on every open set.

Def: A <u>Cartier divisor</u> on a scheme X is given by an open cover $\{U_i\}$ of X together w/ $f_i \in \Gamma(U_i, \mathcal{K}^*)$ such that for each i, j,

$$f_i / f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*).$$

i.e. f_i/f_j is regular on the overlap, and it's a unit. Equivalently, a Cartier divisor is a global section of $\mathcal{K}^*/\mathcal{O}^*$ (w/ multiplication).

A Cartier divisor is <u>principal</u> if it is in the image of the map $\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. Two Cartier divisors are <u>linearly equivalent</u> if their difference (i.e. quotient) is principal. The group of Cartier divisors modulo principal divisors is denoted $\mathrm{CaCl}(X)$.

Cartier vs. locally principal vs. Weil

Prop: If X is normal, Noeth., separated, then $D \in \mathrm{Div}\, X$ is locally principal $\Longleftrightarrow$ D is Cartier. The principal Cartier divisors correspond to the principal Weil divisors.

Pf: We've already seen that locally principal $\Rightarrow$ Cartier.

For the other direction, let

$$\{U_i\},\ f_i \in \Gamma(U_i, \mathcal{K}^*) = K(X)^*$$

be the data of a Cartier divisor on X. We define the associated Weil divisor as follows:

For each prime divisor $Y \subseteq X$, take U_i s.t. $U_i \cap Y \neq \emptyset$. Take the coefficient of Y to be $v(f_i)$. If U_j also intersects Y, then $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$, so $v_Y(f_i) = v(f_j)$, so the coefficient is well-defined, and we get a well-defined Weil divisor, $D = \sum v_Y(f_i)\, Y$.

Note that $D \cap U_i = (f_i)$ by construction, so D is locally principal.

The principal Cartier divisors are exactly those that come from $f \in K^*$. i.e. they are the principal Weil divisors. □

Thus, w/ X as in the prop, we have

$$\{\text{principal divisors}\} \subseteq \{\text{loc. principal divisors}\} \subseteq \{\text{Weil divisors}\}$$

In general, neither inclusion is an equality. For the second inclusion, consider the following example:

Ex: Let $X \subseteq \mathbb{A}^3_k$ be the cone $X = \operatorname{Spec} A$, with

$$A = k[x,y,z]/(xy - z^2).$$ (Recall this is normal)

Recall that $\mathrm{Cl}\,X = \mathbb{Z}/2\mathbb{Z}$ and is generated by Y, the line cut out by $y = z = 0$.

Suppose there is a neighborhood U of the point $\mathfrak{m} = (x,y,z)$ s.t. $Y \cap U$ is principal. Assume WLOG that $U = \operatorname{Spec} B$ is a distinguished open, so that $(y,z)B$ is principal (This is a consequence of something in an earlier proof: We showed that B normal and $Y \subseteq \operatorname{Spec} B$ prime. Then Y principal $\Leftrightarrow$ corr. prime ideal is principal in B) Then, since the origin is in U, we can localize, and get

$$(y,z)B_{\mathfrak{m}} = (y,z)A_{\mathfrak{m}} \text{ is principal.}$$

In $A_{\mathfrak{m}}$, $\mathfrak{m}/\mathfrak{m}^2$ is a 3-dim vector space, generated by $\bar{x}, \bar{y}, \bar{z}$, so the image of $(y,z)A_{\mathfrak{m}}$ is 2-dim, and thus can't be principal (Do you see why?), which is a contradiction.

Thus Y is <u>not</u> locally principal, so in particular, $\mathrm{CaCl}(X) = 0 \neq \mathbb{Z}/2\mathbb{Z} = \mathrm{Cl}(X)$.

If, in addition X is "locally factorial", i.e. all local rings are UFDs, then all Weil divisors are Cartier divisors. That is:

Theorem: Let X be integral, separated, and Noetherian, such that all local rings are UFDs.* Then the group of Weil divisors is naturally isomorphic to the group of Cartier divisors. Thus, $\mathrm{CaCl}\,X \cong \mathrm{Cl}\,X$.

* e.g. nonsingular X satisfy this

Pf: First note that UFDs are normal, so the Cartier divisors are exactly the locally principal divisors. Thus, we just need to show that all Weil divisors are locally principal.

Let D be a Weil divisor and $x \in X$ any point. Then D induces a Weil divisor D_x on $\operatorname{Spec} \mathcal{O}_x$.

(Take $U = \operatorname{Spec} A$ an affine nhood of x. Then
$A \to A_x \cong \mathcal{O}_x$, so D_x is the preimage of D in
$\operatorname{Spec} \mathcal{O}_x \to \operatorname{Spec} A$.)

$\mathcal{O}_x$ is a UFD, so D_x is principal (as discussed earlier), so let $D_x = (f_x)$, $f_x \in K^*$, where $K = K(X)$.

Then D and D_x agree at x. That is, they differ by a divisor $\sum a_i Y_i$ where $x \notin Y_i$ for any i. In particular, they agree on an open set U_x.

Thus $\{(U_x, f_x)\}$ define a locally principal divisor. This construction is the inverse of the conversion loc. princ. div. $\longrightarrow$ Weil div., so we get a 1-1 correspondence. $\square$

Effective Cartier divisors

If X is normal, which locally principal divisors are effective? Given $\{(U_i, f_i)\}$ locally principal, it is effective $\Leftrightarrow$ (f_i) is effective on U for each i.

$$(f_i) \text{ effective} \Leftrightarrow (f_i) \cap \operatorname{Spec} A \text{ is effective } \forall \text{ affine opens } \underset{U}{\overset{\operatorname{Spec} A}{\cap}}$$

$$\Leftrightarrow f \in \bigcap_{\substack{\operatorname{codim} p \\ = 1}} A_p = A.$$

$$\Leftrightarrow f \in \mathcal{O}(U).$$

We can use this characterization to define effective Cartier divisors on an arbitrary scheme:

Def: A Cartier divisor D is <u>effective</u> if it can be written $\{(U_i, f_i)\}$ where each $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. We write $D \geq 0$.